Ottawa Team Math Contest

Team Round Solutions

Part A: Each correct answer is worth 3.

1. What is the last digit of the product of the first 2025 odd prime numbers?

   A) 1 B) 3 C) 5 D) 7 E) 9

   5 is multiplied. Odd multiples of 5 always have last digit 5.

   ~Problem proposed by Rex

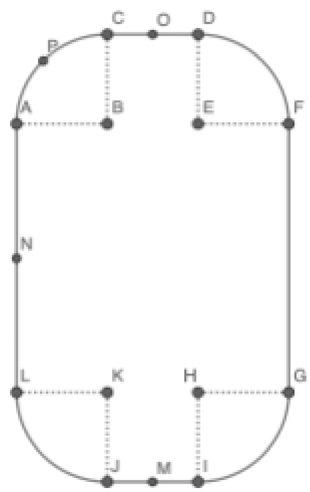
2. Suppose Eric adds up all the digits of the positive integer 3^{2025}, getting the positive integer n_1. Then, he adds up all the digits of n_1 to get n_2. He keeps repeating this process until he ends up with a one-digit number. What is that number?

   A) 1 B) 3 C) 5 D) 6 E) 9

   A number that is a multiple of 9 always has its sum of digits a multiple of 9.

   ~Problem proposed by Rex

3. A hockey rink (as shown) is defined by a 12-sided polygon ABCDEFGHIJKL where angles measures of B, E, H, K are 270º, AL = FG = 3, the remaining lengths are 1, and four quarter circles join the corners to complete the shape. Let a puck rebound off the perimeter of ACDFGIJL such that its incident angle is equivalent to its rebounded angle. Given the points M, N, O, P, find what point puck P just rebounded off of, given it started at point M in the direction of point N and has already rebounded 2025 times.



A) M B) N C) O D) P E) None of the above

2025 == 1 (mod 4) → Puck moves once, M goes to N.

~Problem proposed by Karan

4. What is the remainder when $6^6 + 7^7 + 8^8$ is divided by 10?

A) 5 B) 6 C) 7 D) 8 E) 9

Find Patterns of Last Digit:
Powers of 6 always have the last digit 6.
Powers of 7: 7, 9, 3, 1, 7, 9, 3
Powers of 8: 8, 4, 2, 6, 8, 4, 2, 6
→ 6 + 3 + 6 = 15

~Problem proposed by Karan

5. BattleCatsBoy is trying to build a deck of 8 cards, and each card in this game ranges from 1 to 9 elixirs, inclusive (For those who play Clash Royale, "The Mirror" is not considered as a card here). The application calculates the deck's average elixir cost by summing up all individual card's elixir costs and dividing it by 8 - rounding it to the nearest 1 decimal place. Given that there are exactly 6 one-elixir cards in the game, and unlimited number of other cards, what is the average elixir cost of the 13th lowest elixir deck one can build; in other words, what is the 13th lowest possible average elixir cost in the game?

   A) 2.4 B) 2.5 C) 2.6 D) 2.7 E) 2.8

   The lowest total elixir cost is 1+1+1+1+1+1+2+2 = 10. (there are 6 one elixir cards so the optimal way is to use all 6 of them and fill in the rest with 2 elixir cards)

   Each place lower would have the total elixir cost increased by 1 from the previous place (i.e. 2nd lowest would have total 11, 3rd would have 12 etc.), then the 13th has a total cost of 22.
   22/8 = 2.75 → 2.8

   ~Problem proposed by Rex

6. There are 5 lines in a plane, and 3 of these lines are concurrent to each other (i.e. they intersect at one point), and the rest are not. None of the lines are parallel to each other. How many points of intersections are there?

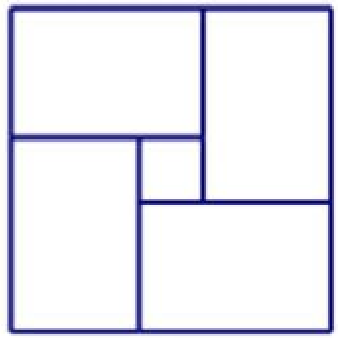
   A) 6 B) 8 C) 10 D) 12 E) 14

   If no lines are concurrent, then they would have C(5, 2) = 10 intersections.
   However, given that 3 lines are concurrent, which means they would've intersected at C(3,2) = 3 points but now only 1 point, the total is subtracted by 2: 10 - 2 = 8

   Note: The number of intersections of n lines given no lines are parallel or concurrent can be written as C(n, 2) because every possible combination of 2 lines yields one intersection.

   ~Problem proposed by Rex

7. A large square is divided into 4 congruent rectangles and a smaller square as shown. Given that each rectangle has area 35 and the small square has area 4, find the positive difference between the length and the width of each rectangle.



A) 1 B) 2 C) 3 D) 4 E) 5

The total area is 35 x 4 + 4 = 144. Thus, the side length of the large square is 12. Label the long side of a congruent rectangle as x, and the short side as y. Therefore, x+y = 12 and xy = 35. Solving this equates x = 7 and y = 5. Thus, x-y = 2.

~Problem proposed by Rex - Inspired by 2016, the 27th Shanghai Yatai Tournament Mock Final (Grade 4) Problem #2

8. Three couples each send text messages on a fixed schedule between 8:00 AM and 9:00 AM—which we treat as minute 0 through minute 60:
   - Karan & Vickie: First text at minute 0, then every 6 minutes (0,6,12…)
   - Matthew & Emily: First text at minute 1, then every 7 minutes
   - Eric & Juliann: First text at minute 3, then every 11 minutes

   A text event is a minute in which at least one couple sends a text. How many total text events occur between minutes 0 and 60 (inclusive)?

A) 21 B) 22 C) 23 D) 24 E) 25

Karan & Vickie text 11 times in the hour. Matthew & Emily text 9 times in the hour. Eric & Juliann text 6 times in the hour. However, there are double instances, namely at 8:36, where all three couples text at the same time. Therefore, the answer is 11 + 9 + 6 - 2 = 24

~ Problem proposed by Rex, the only single person in OTMaC 2025 Team at the time

9. $x + y + z = 15,\ xy + yz + xz = 74$. Compute $x^2 + y^2 + z^2$.

A) 57 B) 67 C) 77 D) 87 E) 97

$x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy + yz + xz) = 15^2 - 2(74) = 225 - 148 = 77$

~ Problem proposed by Rex

10. Define a 45º or 30º measure angle as a “special angle”. How many unique **scalene** triangles are there such that all 3 of the angles have a measure of integer degree and have at least one special angle?

A) 138 B) 140 C) 141 D) 284 E) 285

Suppose a scalene triangle with one angle being 30. The other two angles must sum to 150. Example combinations include 1 and 149, 2 and 148 …… 74 and 76. Thus, it is evident that there are 73 unique scalene triangles with one angle being 30 (you must exclude the case of 30-30-120 and 30-75-75 since those aren't scalene). Similarly, there are 66 unique triangles that are scalene with an angle of 45. However, we must eliminate the double case of a 30, 45, triangle which is included in both settings. Thus, the answer is 73 + 66 - 1 =138

~Problem proposed by Karan

Part B: Each correct answer is worth 4.

11. The nine squares of a 3 × 3 grid are to be colored red, white, and blue, so that no row or column contains squares of the same color. How many different patterns can be made?

A) 4 B) 6 C) 9 D) 12 E) 18

Name the 3 colours A, B, C. It is obvious that there are only 2 triples of combinations of A, B, C such that if we fill in the 3 rows with them (each row with a triple) by any order, no column contains squares of the same colour.
Triple 1:
ABC
BCA
CAB
Triple 2 (you may treat it as a horizontal reflection of triple 1):
CBA
ACB
BAC
Each of the above cases can be rearranged in P(3,3) = 6 possible permutations.
Therefore, total = 2 x 6 = 12

~Problem proposed by Rex - Taken from AMSP 2023 Counting Strategies Level 2 Course

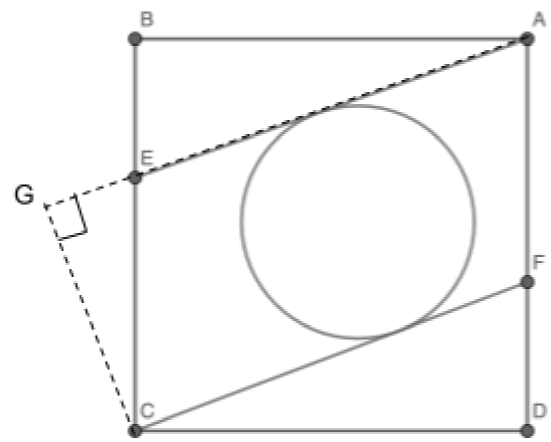
12. Let S be the value of the following expression:
$(1+\frac{2}{1})(1+\frac{2}{2})(1+\frac{2}{3})(1+\frac{2}{4}) \bullet\bullet\bullet (1+\frac{2}{2024})(1+\frac{2}{2025})$
What is the sum of all digits of S?

A) 19 B) 20 C) 21 D) 22 E) 23

Add the fractions inside the bracket and telescope yields:
$\frac{3}{1} \bullet \frac{4}{2} \bullet \frac{5}{3} \bullet \frac{6}{4} \bullet \ldots\ldots \bullet \frac{2026}{2024} \bullet \frac{2027}{2025} = \frac{2026 \times 2027}{1 \times 2}$
From here, you can either brute force the multiplication or simply just realize that: 2026 == 1 and 2027 == 2 (mod 9), which means answer == $\frac{1 \times 2}{1 \times 2} = 1$ *(mod 9). Only 19 in the choices satisfies 1 mod 9 so the answer must be 19.*

~Problem proposed by Rex - Inspired by 2016, the 27th Shanghai Yatai Tournament Mock Final (Grade 4) Problem #4

13. Let ABCD be a square with area 10. Point E is on line segment BC such that CE = 2BE; point F is on line segment AD such that AF = 2DF. A circle is drawn so that it is tangent to AE and CF, what is the area of that circle?



A) $\frac{2}{3}\sqrt{10}\pi$ B) $\frac{3}{\sqrt{10}}\pi$ C) π D) $\frac{\sqrt{10}}{3}\pi$ E) $\frac{10}{9}\pi$

The square has side length of $\sqrt{10}$. Construct line segment CG such that CG is perpendicular to line AE at G, as shown. It is obvious that triangles ABE and CGE are similar, both having side length ratio 1-3-$\sqrt{10}$. Since $CE = \frac{2}{3}BC = \frac{2\sqrt{10}}{3} \Rightarrow CG = 2$, The radius of the circle is just half of CG, which is 1. Thus, the area is π.

~Problem proposed by Rex

14. Some positive integers sum up to 11 and their maximum possible product is P; some other positive integers have a product of 80 and their minimum possible sum is S. What is P - S?

A) 17 B) 32 C) 35 D) 41 E) 43

With some basic experimentation, 3 + 3 + 3 + 2 = 11 gives the largest possible product of 54. Similarly, 4 + 4 + 5 or 2 + 2 + 2 + 2 + 5 = 13 gives the minimum possible sum while having a product of 80. Therefore, the solution is 54 - 13 = 41

~Problem proposed by Rex

15. There are unique integers a_2, a_3, a_4, a_5 such that $\frac{4}{5} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!}$ subject to $0 \leq a_i < i$ for $i = 2, 3, 4, 5$. Compute $a_2 + a_3 + a_4 + a_5$

A) 6 B) 7 C) 8 D) 9 E) 10

Make all the denominators 5! = 120, then you would get $96 = 60a_2 + 20a_3 + 5a_4 + a_5$
With the constraints given, it is obvious that only $a_2, a_3, a_4, a_5 = 1, 1, 3, 1$ *work*

~Problem proposed by Rex - Inspired by 1999 AHSME Problem #25

16. Let ABCD be a trapezoid where AB || CD and AB > CD. Let the angle bisector of B intersect AD at E such that E lies on the circumcircle of △ABC. Given that ∠ABE = 15°, ∠ADC = 90°, and DC = 2. The length of BE can be represented as $a(\sqrt{b} + \sqrt{c})$ for some positive integers a, b, and c, where b and c do not have any perfect square divisors other than 1.

What is a + b + c?

A) 8 B) 9 C) 10 D) 11 E) 12

∠ABE = 15° and BE bisects ∠B implies that ∠ABC = 30°. Since E lies on the circumcircle of △ABC (aka. A, B, C, E cyclic), ∠AEC = 180° - ∠ABC = 150°. Then, ∠CED = 30°, yielding CE = 4 as △CED is 30-60-90. It is also obvious that ∠BCE = 90° as ∠BCD = 180° - ∠ABC = 150° (AB || CD) and ∠DCE = 60°. As it is found that △BCE is a 15-75-90 triangle with CE = 4, one may use the well known result that sin(15°) = $\frac{\sqrt{6}-\sqrt{2}}{4}$ *from here to directly find BE =* $4(\sqrt{6} + \sqrt{2})$*, or this will result in a few more extra steps. For example, it is easy to see that △ABE ≡ △CBE (AAS), yielding AE = CE = 4, then line segment CF may be constructed such that F is on AB and AD || CF (aka. CF is the translation of AD by 2 units to the right), so CF = AD = AE + DE = 4 +* $2\sqrt{3}$ *(note that DE =* $2\sqrt{3}$ *because of the earlier result that △CED is 30-60-90. Then knowing that*
△BCF is also 30-60-90 yields BC = 2CF = 8 + $4\sqrt{3}$*, which also implies that AB = 8 +* $4\sqrt{3}$*. Now, applying Pythagorean Theorem in △ABE, using that AE = 4, AB = 8 +* $4\sqrt{3}$*, is sufficient to find that BE =* $4(\sqrt{6} + \sqrt{2})$*.*

~Problem proposed by Karan - Approved by Professor Dragos Calitoiu

17. Rex wants to play PUBG with Karan, and they decided to meet at McDonald's this morning. Rex really wants to play PUBG but sleeps too much sometimes, so he can arrive any time at McDonald's between 9am and 11am and he can stay for at most 2 hours but he must leave no later than 12pm. Karan says he has to do robotics, so he will leave immediately if he doesn't see Rex at McDonald's the moment he arrives, and Karan can arrive any time between 9am and 12pm. If the probability of Rex successfully meeting Karan can be expressed as $\frac{m}{n}$ such that m and n are coprime, what is m + n?

A) 7 B) 9 C) 13 D) 17 E) 19

We may casework this problem by first considering the time range Karan arrives.

Case 1: *Karan arrives between 9am and 11am, the same time range as Rex (P =* $\frac{2}{3}$*)*

In this case, as long as Rex arrives before Karan, they get to play together, and there is $\frac{1}{2}$ *chance of that happening by intuition. So Case 1:* $\frac{2}{3} \bullet \frac{1}{2} = \frac{1}{3}$

Case 2: *Karan arrives between 11am and 12pm (P =* $\frac{1}{3}$*)*

This case is a bit more complicated as caseworking Rex's arrival time is also required.

> ***Case 2.1:*** *Rex arrives between 10am and 11am (P =* $\frac{1}{2}$*)*
>
> *If Rex arrives in this time range, which means he wouldn't leave until 12, then it is obvious that he can 100% meet Karan successfully. So Case 2.1:* $\frac{1}{3} \bullet \frac{1}{2} = \frac{1}{6}$
>
> ***Case 2.2:*** *Rex arrives between 9am and 10am (P =* $\frac{1}{2}$*)*
>
> *Suppose Rex arrives x minutes past 9 and Karan arrives y minutes past 11 for* $0 \leq x, y \leq 60$*. Then by intuition, they can meet successfully if and only if* $x > y$*, which there is a 50% chance. So Case 2.2:* $\frac{1}{3} \bullet \frac{1}{2} \bullet 50\% = \frac{1}{12}$

Overall probability = P(Case 1) + P(Case 2.1) + P(Case 2.2) = $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} = \frac{7}{12}$

Answer = 7 + 12 = 19

~Problem proposed by Rex - Inspired by real world context

18. Let I be the intersection of two chords AB and CD, where BI > DI. Take D' on BI such that DI = DD'. Given AI = 2, CI = 3, D'I = $2\sqrt{6} - 2\sqrt{2}$, and ∠DD'I = 75º, Find BI.

A) 5 B) 6 C) 7 D) 8 E) 9

In triangle DD'I, we can use the cosine law using angle D'DI (180 - 2(75) = 30) and D'I (given). From this, we find that DI = DD' = 4. Now, we can apply the Power of a Point of I where CI x DI = AI x BI. From this, we conclude BI = 6.

~Problem proposed by Karan

19. If a Baby Dragon weighs 100kg, an Ice Golem weighs 85kg, a Hog weighs 70kg, and there are exactly X Baby Dragons, Y ice golems, and Z Hogs that weigh a total of 2025kg (X, Y, Z are nonnegative integers), how many possible triples of (X, Y, Z) are there?

A) 17 B) 18 C) 19 D) 20 E) 21

Since the total weight is odd, there must be an odd number of Ice Golems.
Thus, casework can be done with the cases categorized by the number of Ice Golems.
1 Ice Golem → (# of Baby Dragon, # of Hog) = (4, 22), (11, 12), (18, 2) → 3 triples
3 Ice Golem → (# of Baby Dragon, # of Hog) = (3, 21), (10, 11), (17, 1) → 3 triples
5 Ice Golem → (# of Baby Dragon, # of Hog) = (2, 20), (9, 10), (16, 0) → 3 triples
7 Ice Golem → (# of Baby Dragon, # of Hog) = (1, 19), (8, 9) → 2 triples
9 Ice Golem → (# of Baby Dragon, # of Hog) = (0, 18), (7, 8) → 2 triples
11 Ice Golem → (# of Baby Dragon, # of Hog) = (6, 7) → 1 triple
13 Ice Golem → (# of Baby Dragon, # of Hog) = (5, 6) → 1 triple
15 Ice Golem → (# of Baby Dragon, # of Hog) = (4, 5) → 1 triple
17 Ice Golem → (# of Baby Dragon, # of Hog) = (3, 4) → 1 triple
19 Ice Golem → (# of Baby Dragon, # of Hog) = (2, 3) → 1 triple
21 Ice Golem → (# of Baby Dragon, # of Hog) = (1, 2) → 1 triple
23 Ice Golem → (# of Baby Dragon, # of Hog) = (0, 1) → 1 triple
Total = 20 triples
Note: Although this method seems very tedious, some patterns can be easily found as you casework things through if you have good number sense, so the method does not take as long as it seems. In particular, for cases with 11+ Ice Golem, the pattern is obvious.

~Problem proposed by Rex (Fun Fact: Baby Dragon (Rex) actually weighs ~100kg)

20. The only non-integer rational solution to the following polynomial $9x^6 + 12x^5 - 2x^4 - x^3 + 3x^2 - x$ can be represented as a/b, where a and b are coprime integers. What is |100a - b|?

    A) 97 B) 197 C) 297 D) 397 E) 497

    Using Polynomial Remainder and Factor Theorem, we know that possible roots of this polynomial are 1, -1, 1/3, -1/3, 1/9, -1/9. With some computation and intuition, the rational solution is 1/3. Thus, |100a - b| is 97.

    ~Problem proposed by Karan

Part C: Each correct answer is worth 6.

21. Triangle ABC has a right angle at B and contains a point P such that PA = 10, PB = 6, and ∠APB = ∠BPC = ∠CPA. Find the length of line segment PC.

33

Let PC be x.
From ∠APB = 120º (360º/3) with PA = 10 and PB = 6, the Law of Cosines gives:
$AB^2 = 10^2 + 6^2 - 2 \bullet 10 \bullet 6 \bullet cos(120^o) = 100 + 36 - 2 \bullet 10 \bullet 6 \bullet (-\frac{1}{2}) = 196$
Using the same logic, ∠BPC = 120º (360º/3) with PB = 6 and PC = x gives
$BC^2 = 36 + x^2 + 6x$
Similarly, ∠BPC = 120º (360º/3) with PB = 10 and PC = x gives
$AC^2 = 100 + x^2 + 10x$
Since ΔABC *is right-angled at B, by Pythagorean Theorem,* $AB^2 + BC^2 = AC^2$
Substituting the values yields x = 33.

~Problem proposed by Rex - Taken from AMSP 2023 Computational Geometry Course

22. If integers a, b, c, d form a geometric sequence such that 0 < a < b < c < d < 1000, how many possible quadruples of (a, b, c, d) are there?

286

A sufficient way to count them is to observe that any four-term integer geometric sequence (a, b, c, d) *with* $0 < a < b < c < d < 1000$ *must arise from choosing positive integers* $p > q$ *with gcd(p, q) = 1 and a positive integer k, then setting:*

$$a = q^3k,\ b = q^2pk,\ c = qp^2k,\ d = p^3k$$

Hence counting all valid quadruples reduces to:

1. *Pick p = 2, 3, …, 9 (we need* $p^3 < 1000$*.)*
2. *Pick q with* $1 \le q < p$ *and gcd(p, q) = 1.*
3. *Pick k so that* $q^3k < 1000$*.*
4. *Each such triple (p, q, k) gives a unique quadruple (a, b, c, d)*

A quick tally:

- *For p = 2,* $p^3 = 8$*. There is only q = 1, and [999/8] = 124 choices of k.*
- *For p = 3,* $p^3 = 27$*. Valid q = 1, 2 (both coprime to 3). Each allows [999/37] = 37 choices of k.*
- *For p = 4,* $p^3 = 64$*. Coprime* $q \in \{1, 3\}$*. Each has [999/64] = 15 choices of k.*
- *For p = 5,* $p^3 = 125$*. Coprime* $q \in \{1, 2, 3, 4\}$*. Each has [999/125] = 7 choices of k.*
- *For p = 6,* $p^3 = 216$*. Coprime* $q \in \{1, 5\}$*. Each has [999/216] = 4 choices of k.*
- *For p = 7,* $p^3 = 343$*. Coprime* $q \in \{1, 2, 3, 4, 5, 6\}$*. Each has [999/343] = 2 choices.*
- *For p = 8,* $p^3 = 512$*. Coprime* $q \in \{1, 3, 5, 7\}$*. Each has [999/512] = 1 choice.*
- *For p = 9,* $p^3 = 729$*. Coprime* $q \in \{1, 2, 4, 5, 7, 8\}$*. Each has [999/729] = 1 choice.*

Summing all these contributions gives
p = 2: 124
p = 3: 2 x 37 = 74
p = 4: 2 x 15 = 30
p = 5: 4 x 7 = 28
p = 6: 2 x 4 = 8
p = 7: 6 x 2 = 12
p = 8: 4 x 1 = 4
p = 9: 6 x 1 = 6

$\Rightarrow Total = 124 + 74 + 30 + 28 + 8 + 12 + 4 + 6 = 286$

~Problem proposed by Eric

23. A number set S contains 7 elements: S = {2, 3, 4, 6, 9, 12, 18}. If the product of a set is the product of all the elements in it, how many subsets of S (including S and empty set) has a product that is a multiple of 36?

101

Observing that every element in S is either a multiple of 2 or multiple of 3 or both, and 2 and 4 are the only two elements that is only a multiple of 2 AND 3 and 9 are the only two elements that is only a multiple of 3, we can conclude that if a subset of S has 4 elements or more, then its product must be a multiple of 36.
In total there are 2^7 = 128 subsets of S, and there are 128/2 = 64 subsets with 4 or more elements (subsets with 4 or more elements are symmetrical/opposite to 3 or less for a set of 7 elements). Subsets with 3 or less elements can be explored using casework.
Case 1: *Length = 3*
It can be observed that most subsets with length 3 work, so subtracting those that don't work from C(7,3) = 35 would yield the number of subsets with length 3 that is a multiple of 36.
Subsets that ***do not*** *work contain either both 2,4 or both 3,9:*
{2,3,4}, {2,4,6}, {2,4,12}, {2,3,9}, {3,6,9}, {3,9,18}
Therefore, there are 35 - 6 = 29 subsets of length 3 that is a multiple of 36.
Case 2: *Length = 2*
Unlike Case 1, most subsets do not work here, so simply listing out the ones that work would be the strategy here. One way of doing this is to start with 18, the greatest number in S, then go lower.
{18, 2}, {18, 4}, {18, 6}, {18, 12}
{12, 3}, {12, 6}, {12, 9}
{9, 4}
Total = 8 subsets.

Therefore, combining all the subsets, there are a total of 64 + 29 + 8 = 101 subsets that are multiples of 36.

~Problem proposed by Rex - Inspired by 2022 CIMC Problem A6

24. What is the sum of all positive divisors of 2025^2 that has a last digit of 1 OR 5?

7682623

*First count all the divisors that have last digits 1. Since $2025^2 = 3^8\, 5^4$, such divisors (divisors with last digit 1) may only be written as 3^a for some integers $a \in [0, 8]$ as such divisors must **not** be multiples of 5. By finding patterns, the divisors only have the last digit 1 for a = 0, 4, 8, which corresponds to 1, 81, 6561, respectively.*

Then, count all the divisors that have last digits 5, which are simply just the divisors that are multiples of 5, since all divisors of 2025^2 are odd (as 2025^2 itself is odd). Therefore, since such divisor is expressed as $3^a\, 5^b$ for some integers $a \in [0, 8]$, $b \in [1, 4]$, the sum of all such possible divisors is expressed as $(3^0 + 3^1 + 3^2 + \ldots + 3^8)(5^1 + 5^2 + 5^3 + 5^4)$. This is because the expansion of the product above has every possible divisor as a unique term (there is a specific topic on finding the number or sum of divisors of a number using its prime factorization: https://en.wikipedia.org/wiki/Divisor_function)
Using geometric sequence summation may simplify the calculation inside each bracket, which yields 9841 x 780 = 7675980

Total = 7675980 + 6561 + 81 + 1 = 7682623

~Problem proposed by Rex - Inspired by 2025 HMMT Number Theory Round Problem 1

25. In Clash Royale's Path of Legends, players gain 1 step for a win and lose 1 step for a loss, unless they are on a Golden Step, which cannot be lost. A Golden Step is earned after winning 3 games (not necessarily consecutively) since the last Golden Step. Once earned, it remains permanently. Rex starts at step 0 (a Golden Step) and needs to reach step 10 to enter Challenger III. Given that Rex has a 80% probability of winning each game, what is the probability that he reaches Challenger III within 12 games? Round to 2 significant digits.

Useful Fact: $0.8^{10} \approx 0.1074$

Example Scenario:

- Rex starts at step 0 (a Golden Step).
- He loses his first 10 games but stays at step 0 because step 0 is a golden step.
- He then wins 2 games (reaching step 2) but loses the next game (dropping to step 1).
- Winning another game moves him back to step 2, and since he has now won 3 games total since step 0, step 2 becomes a Golden Step.
- From here, he can never drop below step 2, no matter how many times he loses.
- The process repeats: every time Rex accumulates 3 wins after the previous Golden Step, he earns a new one.

0.34

For formatting reasons, the solution is written on the next page.

*Since this problem only considers scenarios where Rex reaches Challenger III within **12** games, which means the most number of wins Rex gets in total is 11 as before he reaches Challenger III (he can't just have 12 wins because then he would be over step 10, which is not what is considered here). As Rex also must win a minimum of 10 games, Rex will encounter exactly 4 golden steps in total, including the one at step 0, because he wins a total of either 10 or 11 games.*

Case 1: Rex wins 10 games and never loses.
This is the most trivial case as the golden step doesn't even matter here. The probability of this happening is simply just $0.8^{10} = 0.1074$ *(this result is also given in the problem).*
Case 2: Rex wins 10 games and loses 1 game on a golden step.
Since the loss is on a golden step, it does not add much complexity to the scenario. Simply consider Case 1 where Rex wins all 10 games, which results in golden steps at step 0, 3, 6, 9 (the exact golden step positions in fact do not matter since it is found that there is always a total of 4 golden steps, but they are listed out here for better understanding). Then, in this case, simply insert Rex's only loss on one of the 4 golden steps. Then this becomes $0.8^{10} \times 0.2^{1} \times 4$ *as the probability of losing that game is 0.2, and there are 4 positions to put that loss, which results in* $0.8^{11} = 0.8 \times 0.1074 = 0.0859$
Case 3: Rex wins 10 games and loses 2 games on golden steps
Similar logic to Case 2 but this time 2 losses are inserted at golden steps. Since there are 4 golden steps available and 2 losses to insert, one may simply use C(4,2) = 6, but keep in mind that 2 losses can also be inserted into the same golden step as Rex can lose twice in a row on a golden step (it would be a skill issue at this point, but definitely possible). Thus, we need to add 4 more cases, for a total of 10. Then this becomes: $0.8^{10} \times 0.2^{2} \times 10 = 0.0429$
Case 4: Rex wins 11 games and loses 1 game on a non-golden step.
This is the most complicated case as it can be tricky to determine the number of possible positions of the loss. Using the same logic as other cases, we may still let Rex win 11 games first, then try to insert the loss, but this time it is crucial to consider special cases. Since this time Rex is not losing on a golden step, it is obvious that Rex loses after the n*th win for* $n \equiv \{1, 2\} \pmod 3$*, so n = {1, 2, 4, 5, 7, 8, 10, 11}. But keep in mind that n actually can not be 10 or 11 since Rex would've already reached Challenger III at that point, so there are only 6 valid choices for n. Then, it becomes* $0.8^{11} \times 0.2^{1} \times 6 = 0.1031$

Now, adding up all the cases gives: 0.1074 + 0.0859 + 0.0429 + 0.1031 = 0.34 (2 s.f.)
Due to potential rounding errors & inconsistencies, 0.33 would also be accepted.

~Problem proposed by Rex